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ALGEBRAIC GEOMETRY OF THE THREE-STATE CHIRAL POTTS MODEL

BY

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ABSTRACT

For more than a decade now, the chiral Potts model in statistical mechanics has attracted much attention. A number of mathematical physicists have written quite extensively about it. The solutions give rise to a curve over $\mathbb{C}$, and much effort has gone into studying the curve and its Jacobian.

In this article, we give yet another approach to this celebrated problem. We restrict attention to the three-state case, which is simplest. For the first time in its history, we study the model with the tools of modern algebraic geometry. Aside from simplifying and explaining the previous results on the periods and Theta function of this curve, we obtain a far more complete description of the Jacobian.

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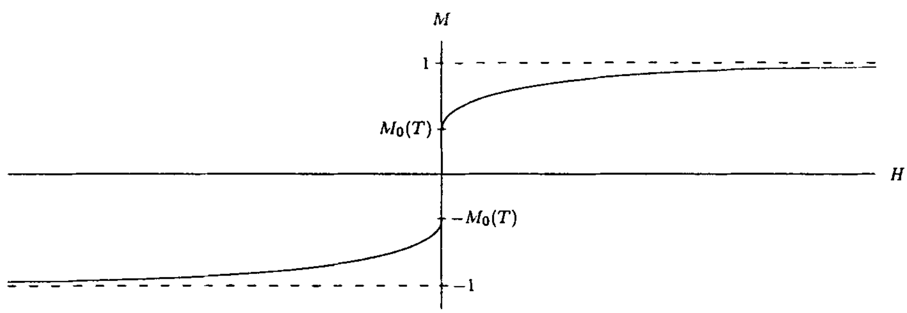
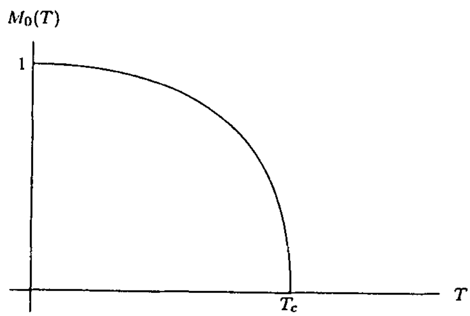
1. Introduction

Suppose we take a ferromagnet and put it in a magnetic field. Then the ferromagnet will magnetise. Let H be the external magnetic field, T the temperature, and $M(H,T)$ the magnetisation of the ferromagnet. The graph of $M(H,T)$ as a function of H, with T held constant, looks approximately like this:



The key point is that, as $H \longrightarrow 0$, the magnetisation $M(H,T)$ tends to some non-zero number, denoted $M_0(T)$. It is the spontaneous magnetisation of the ferromagnet, at temperature T.

Next, one can wonder how $M_0(T)$ varies, as a function of T. One knows experimentally that its graph looks approximately like



The spontaneous magnetisation drops off, until at some temperature T_c the material stops being a ferromagnet.

A mathematical model that simply explains this phenomenon is the Ising model. The Ising model is that the particles lie on a lattice, and each is capable of two states. Adjacent particles contribute to the energy of the system, and the contribution depends only on whether they are in the same state, or in different states. The model was formulated and solved, in one dimension, by Ising in 1925. The 2-dimensional solution was due to Onsager and Yang, in the 1940's. The 3-dimensional problem is still unsolved.

Solving the 2-dimensional Ising model amounts to finding the eigenvalues of some large square matrix. The solution of Onsager and Yang, later elaborated and extended by Baxter and many others, goes as follows. Instead of one matrix, one introduces a large family of commuting matrices. The matrices are parametrised by points on some elliptic curve. The thermodynamics of the system is now parametrised by the elliptic curve, and we can use Theta functions to explicitly exhibit how the various thermodynamic observables depend on the point in the curve. Clever manipulation with Theta functions now solves the problem.

The chiral Potts model is the generalisation of the Ising model, where more states are allowed. Instead of each particle being in one of only two possible states, we now allow N states. The problem has been to generalise the existing methods to this new model. And the difficulty is that, instead of an elliptic curve, one discovers oneself faced with a curve of higher genus.

The curve is very explicit. It comes from a solution to the the star-triangle relations; see [9]. Specifically, let N be the number of states, k the temperature (suitably normalised). Let $k' = \sqrt{1-k^2}$. The Boltzmann weights depend on four complex numbers a, b, c, d which must satisfy

$$(1) \qquad \begin{bmatrix} 1 & k' & 0 & -k \\ k' & 1 & -k & 0 \\ 0 & -k & 1 & -k' \\ -k & 0 & -k' & 1 \end{bmatrix} \begin{bmatrix} a^N \\ b^N \\ c^N \\ d^N \end{bmatrix} = 0.$$

Of course, this is of physical significance only when k is real, and in fact the interesting range is $0 < k < 1$. But there is nothing to stop us from studying this family of curves for all pairs of complex numbers (k, k') with

$$k^2 + {k'}^2 = 1, \quad k \neq 0 \neq k'.$$

Due to this last condition the matrix has rank two, so the equations determine two independent hypersurfaces in $\mathbb{P}^3$.

Introducing the standard (see Baxter's [4]) variables $x = a/d$, $y = b/c$, $\mu = d/c$, we have that x and y satisfy the equation

$$x^N + y^N = k(1 + x^N y^N), \tag{2}$$

giving an algebraic curve in $\mathbb{A}^2$. And μ satisfies

$$\mu^N = k'/(1 - kx^N) = (1 - ky^N)/k'.$$

The curve (1) is an N-sheeted unramified cover of the curve (2). The genus of the curve (2) is $(N-1)^2$, while the genus of the curve (1) is $N^2(N-2)+1$. See [12]. In this article, we will restrict ourselves to the very simplest curve. We will look only at the curve (2), and only in the case $N = 3$.

Already here, the physical problem is completely unsolved. And following the classical methods, which originated with Onsager and Yang, one would like to parametrise the physics in terms of Theta functions. It is reasonable to study the Jacobian and the Theta divisor of the curve.

The physicists have worked hard on this problem. We refer the reader particularly to the articles by Matveev and Smirnov [12], by Baxter [3], [4], and [5], and by Davies [10]. In fact these papers are admirable pieces of classical mathematics. The authors want to understand the Jacobian of a curve, and do so very explicitly. They first choose a basis for the homology of the curve, compute the period integrals, and work out identities of Theta functions. Of course, the aim remains to use the algebraic geometry of the curve to obtain results about the physics of the model. But given the effort, industry and ingenuity that the physicists have put into studying these curves, a new treatment of the curves should be of interest, in its own right.

The model remains unsolved, despite all the efforts so far. The frustrations are summarized by recent comments of McCoy (one of the discoverers of the chiral Potts model). In his 1999 Heinemann prize address, McCoy said

> This is a classic problem in algebraic geometry for which in fact no explicit answer is known either. Indeed, the unsolved problems arising from the chiral Potts model are so resistant to all known mathematics that I have reduced my frustration to the following epigram:
>
> *"The nineteenth century saw many brilliant creations of the human mind. Among them are algebraic geometry and Marxism. In the late twentieth century Marxism has been shown to be incapable of*

solving any practical problem but we still do not know about algebraic geometry."

Now as McCoy says, algebraic geometry is one of the brilliant achievements of the late 19^{th} century, and in the present century it saw some major developments. The aim of the current article is to apply these to the problem at hand. [We made no attempt to harness the theory of Marxism, to work for us in this endeavour.] The reader is strongly encouraged to compare our paper to the earlier ones, particularly Matveev and Smirnov [12] and Davies [10]. Not only is our approach more elegant, but the results are far sharper. This article is probably the first one in the history of this problem which methodically applies the modern machinery of algebraic geometry. The aim of this article is to treat the problem with the modern tools of algebraic geometry, and to write the article in such a way that any physicist can read it. The result is quite long, and therefore we decided not to make it yet longer by suggesting physical applications. That will come in a later paper. This present one is concerned only with the study of the curve, its Jacobian, and the Theta divisor.

If we had written this article as private notes to ourselves, it would be much shorter. We tried to make it readable to both physicists and algebraic geometers. After all, if physicists cannot read it then there is no point to it at all. And at the very least, we must expect the referee to be an algebraic geometer; some expert will presumably check that the mathematics is correct. For this reason we included the very long introduction, which points out the physical relevance of the question, and refers the expert to some (not all) of the earlier literature on the geometry of this curve.

In the remainder of this article, we specialize to a study of the curve X determined by

$$x^3 + y^3 = k(1 + x^3y^3),$$

where $k \notin \{0, \pm 1\}$. It has genus four, and it is easy to show ([10] or [12]) that the space of holomorphic differentials is spanned by

$$X = \frac{x\,dx}{y^2 - ky^2x^3}, \quad Y = \frac{y\,dx}{y^2 - ky^2x^3},$$
$$Z = \frac{xy\,dx}{y^2 - ky^2x^3}, \quad W = \frac{dx}{y^2 - ky^2x^3}.$$

Davies noticed that these differentials satisfy the homogeneous identities

$$\begin{aligned} XY &= ZW, \\ X^3 + Y^3 &= k(Z^3 + W^3). \end{aligned}$$

They give a mapping of X into $\mathbb{P}^3$ (the canonical map). Davies showed that this amounts to an embedding of X in a smooth compactification in $\mathbb{P}^3$. In Section 2, we reprove Davies' results, but more in the way an algebraic geometer would approach the problem. We start with the compact curve in $\mathbb{P}^3$ and show that its embedding is canonical. Mathematical physicists are advised to skip Section 2. The above is a summary of the results, and the proofs we give in Section 2 are written for algebraic geometers.

The remainder of this Introduction is an overview of what follows. In Section 3 we study the automorphism group of X. Much has been written about the automorphisms of the chiral Potts curves; in the present case the vital (previously un-noticed) point is that the group is the product of two copies of the permutation group S_3. That is, $\mathrm{Aut}(X) = S_3 \times S_3$. The remainder of Section 3 is devoted to constructing maps from X to elliptic curves, and hence from the Jacobian $J(X)$ to elliptic curves. We construct these maps and use them to compute the action of $S_3 \times S_3$ on the holomorphic 1-forms on X.

In Section 4 we show that the Jacobian $J(X)$ is isogenous to the product of four elliptic curves, which occur as pairs. There are two pairs, and interchanging them corresponds to switching k and $1/k$. That is, let $E(k)$ be the complex torus $\mathbb{C}/\big(1, \tau(k)\big)$, whose period $\tau(k)$ is a function of k. Then there is a homomorphism π

$$J(X) \xrightarrow{\pi} E(k) \times E(k) \times E(k^{-1}) \times E(k^{-1}),$$

with finite kernel. An elementary way to view this is that the lattice which generates the complex torus $J(X)$ may be constructed as a finite number of translates of the product of the lattices which generate $E(k)$ and $E(k^{-1})$.

The structure of this isogeny is obviously of great interest and is investigated in Sections 4–7. Already in Section 4, we show that all points in the kernel of π are of order 3. That is, $\mathrm{Ker}(\pi)$ is contained in the group $J(X)_3 \subset J(X)$, where $J(X)_3$ is the kernel of multiplication by 3. Since $|J(X)_3| = 3^{2g} = 3^8$, this immediately gives

$$|\,\mathrm{Ker}(\pi)| \leq 3^8.$$

We also prove that the action of $S_3 \times S_3$ takes the kernel of π to itself. See Proposition 4.2 and Corollary 4.3.

It follows that there is an action of $S_3 \times S_3$ on $E(k)^2 \times E(k^{-1})^2$, making the map

$$J(X) \xrightarrow{\pi} E(k)^2 \times E(k^{-1})^2$$

a morphism of $S_3 \times S_3$-spaces. In Section 5, we compute quite explicitly the action, as a representation into $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$.

In Section 6 we consider the effect of π on points of order 3 in $J(X)$. We find a subgroup of 3^4 points in the group $J(X)_3/\operatorname{Ker}(\pi)$, which establishes that

$$|\operatorname{Ker}(\pi)| \leq 3^4.$$

The question of the degree is finally settled in Section 7, where we show that

$$\deg(\pi) = |\operatorname{Ker}(\pi)| = 9.$$

We show this by proving that the pushforward of the Theta divisor is a rational multiple of a divisor of degree 9. In symbols: $\pi_*\Theta = r\Delta$, where $\deg(\Delta) = 9$. Computing degrees, we have

$$|\operatorname{Ker}(\pi)| \cdot \deg(\Theta) = r^4 \deg(\Delta).$$

But Θ is a principal polarization; $\deg(\Theta) = 1$, and we know that $\deg(\Delta) = 9$. This allows us to compute

$$|\operatorname{Ker}(\pi)| = 9r^4,$$

for some rational number r. Since the left hand side is an integer dividing 3^4, we must have $r^4 = 1$.

For any elliptic curve E, the divisor $D(E) \subset E \times E$ is defined to be the sum of the three divisors $(x = 0)$, $(y = 0)$ and $(x + y = 0)$. The divisor $\Delta = \pi_*\Theta$ can be explicitly computed. It may be written in the form

$$\pi_*\Theta = \Big[D\big(E(k)\big) \times E(k^{-1})^2\Big] + \Big[E(k)^2 \times D\big(E(k^{-1})\big)\Big].$$

The expression for $\pi_*\Theta$ given above therefore exhibits it as a sum of six abelian subvarieties. In Section 8, we show that the six may be reduced to four; this means that the Theta function of X may be expanded as a sum of products of four elliptic Theta functions. Matveev and Smirnov discovered such an identity; see [12]. In Section 9, we show that the formula of Matveev and Smirnov is one of an infinite family.

2. The basics

Let $k \in \mathbb{C}$ be a complex number, $k \notin \{0, \pm 1\}$. Consider the curve $X \subset \mathbb{CP}^3$ given by the equations

$$xy = zw, \quad x^3 + y^3 = k(z^3 + w^3). \tag{3}$$

Since the field will always be the complex numbers, we will omit it in the notation. $\mathbb{P}^3$ will stand for $\mathbb{CP}^3$. For most of the article, the number $k \notin \{0, \pm 1\}$ is fixed.

The curve X lies on the quadric surface $xy = zw$, and it is well-known that this surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We have a curve $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, cut out by the equation $x^3 + y^3 = k(z^3 + w^3)$. Let F_1 and F_2 be fibers of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. The embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ is by the complete linear system of the very ample divisor $F_1 + F_2$. The curve $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is given by a cubic polynomial in the variables $\{x, y, z, w\}$; this makes it linearly equivalent to the divisor $3F_1 + 3F_2$.

Next, let us check that X is a smooth curve. This is best checked in affine coordinates. For a point in $\mathbb{P}^3$, one of the variables $\{x, y, z, w\}$ must be non-zero; by symmetry, the four cases are the same. Assume therefore that $w \neq 0$. Dividing by w, we may assume $w = 1$, and the curve becomes

$$\begin{aligned} xy &= z, \\ x^3 + y^3 &= k(z^3 + 1), \end{aligned}$$

that is,

$$x^3 + y^3 = k(1 + x^3 y^3).$$

A point on the curve is singular if it satisfies the polynomial equation above, as well as its partial derivatives with respect to x and y. That is, the singular points of this affine part of X are the solutions of the equations

$$\begin{aligned} x^3 + y^3 &= k(1 + x^3 y^3), \\ y^2 &= kx^3 y^2, \\ x^2 &= kx^2 y^3. \end{aligned}$$

Suppose $x = 0$. Then the equation $y^2 = kx^3y^2$ tells us $y = 0$, and the equation $x^3 + y^3 = k(1 + x^3y^3)$ gives that $k = 0$, which is false. Hence $x \neq 0$, and by symmetry $y \neq 0$; it follows that $x^3 = y^3 = k^{-1}$. Substituting in the equation $x^3 + y^3 = k(1 + x^3y^3)$, we obtain

$$2k^{-1} = k(1 + k^{-2}),$$

which becomes $k^2 - 1 = 0$. Since $k \neq \pm 1$, this is false; hence there is no solution to the three equations, and X is smooth.

Thus we have a smooth curve $X \subset \mathbb{P}^1 \times \mathbb{P}^1$, linearly equivalent to $3F_1 + 3F_2$. The canonical divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ is $K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2F_1 - 2F_2$. The canonical divisor of $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the restriction to X of the divisor $X + K_{\mathbb{P}^1 \times \mathbb{P}^1}$, and this is linearly equivalent to

$$(3F_1 + 3F_2) + (-2F_1 - 2F_2) = F_1 + F_2.$$

In other words, the canonical divisor K_X of X is the divisor of its embedding in $\mathbb{P}^3$. The coordinates $\{x, y, z, w\}$ may be thought of as holomorphic 1-forms on X. The exact sequence of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{array}{ccc} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2F_1 - 2F_2) & \xrightarrow{x^3+y^3-k(z^3+w^3)} & \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(F_1 + F_2) \\ & & \downarrow \\ & & \mathcal{O}_X(F_1 + F_2) \\ & & \downarrow \\ & & 0 \end{array}$$

gives an exact cohomology sequence

$$\begin{array}{ccc} H^0(\mathbb{P}^1 \times \mathbb{P}^1, -2F_1 - 2F_2) & & \\ \downarrow & & \\ H^0(\mathbb{P}^1 \times \mathbb{P}^1, F_1 + F_2) & \longrightarrow & H^0(X, F_1 + F_2) \\ & & \downarrow \\ & & H^1(\mathbb{P}^1 \times \mathbb{P}^1, -2F_1 - 2F_2) \end{array}$$

and since

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, -2F_1 - 2F_2) = 0 = H^1(\mathbb{P}^1 \times \mathbb{P}^1, -2F_1 - 2F_2),$$

the map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, F_1 + F_2) \longrightarrow H^0(X, F_1 + F_2)$$

must be an isomorphism. The sections $\{x, y, z, w\}$ form a basis for the vector space of holomorphic 1-forms on X. The embedding of X into $\mathbb{P}^3$ is the canonical embedding, by the complete linear system of holomorphic 1-forms. The genus of the curve X, which is the dimension of $H^0(X, K_X)$, must be 4.

3. The automorphism group of X

Let ζ be a primitive cube root of 1; that is, $\zeta^3 = 1$, but $\zeta \neq 1$. Consider the automorphisms σ, τ, $\overline{\sigma}$ and $\overline{\tau}$, given by the following formulas:

$$\begin{aligned} \sigma(x, y, z, w) =& (y, x, z, w), \\ \tau(x, y, z, w) =& (\zeta x, \zeta^2 y, z, w), \\ \overline{\sigma}(x, y, z, w) =& (x, y, w, z), \\ \overline{\tau}(x, y, z, w) =& (x, y, \zeta z, \zeta^2 w). \end{aligned}$$

It is very easy to compute that σ and τ commute with $\overline{\sigma}$ and $\overline{\tau}$. One can also easily verify that

$$\sigma^2 = 1, \qquad \overline{\sigma}^2 = 1,$$
$$\tau^3 = 1, \qquad \overline{\tau}^3 = 1,$$
$$\sigma\tau\sigma = \tau^2, \qquad \overline{\sigma\tau\sigma} = \overline{\tau}^2.$$

In other words, the automorphisms σ and τ generate a group isomorphic to S_3, as do the automorphisms $\overline{\sigma}$ and $\overline{\tau}$, and these two S_3's commute. There is a homomorphism from the group $S_3 \times S_3$ to the group of automorphisms of X. It may be checked that the automorphism group is equal to $S_3 \times S_3$ provided that $k^2 \neq -1$; but this plays no role in our discussion here.

Of course, the curve X is a ramified double cover of the curve X/σ. Our first observation is

LEMMA 3.1: *Let σ be the involution above, on the curve X. The curve X/σ is an elliptic curve, and the map $\phi : X \longrightarrow X/\sigma$ is ramified at six points. The 1-form $x - y$ vanishes precisely at the ramification points of ϕ. [Recall that the embedding $X \subset \mathbb{P}^3$ is canonical, so that $\{x, y, z, w\}$ can be thought of as holomorphic 1-forms on X.]*

Proof: The ramification points of the map $X \longrightarrow X/\sigma$ are precisely the fixed points of the involution σ. We have to find all the points $P = (x, y, z, w)$ for which $\sigma P = P$. This means that the points $P = (x, y, z, w)$ and $\sigma P = (y, x, z, w)$ agree in $\mathbb{P}^3$; there must exist a non-zero number $\lambda \in \mathbb{C}$ so that

$$(x, y, z, w) = \lambda(y, x, z, w).$$

This gives four equations

$$\begin{aligned} x &= \lambda y, \\ y &= \lambda x, \\ z &= \lambda z, \\ w &= \lambda w. \end{aligned}$$

The last two equations can be written

$$(\lambda - 1)z = 0 = (\lambda - 1)w.$$

Either $\lambda = 1$, or $z = w = 0$. If $z = w = 0$, the equation $xy = zw$ tells us that either x or y must also vanish. The two cases being symmetric, assume $y = 0$. But then the equation

$$x^3 + y^3 = k(z^3 + w^3)$$

reduces to $x^3 = 0$, giving $x = y = z = w = 0$. There is no non-trivial solution with $\lambda \neq 1$.

Therefore $\lambda = 1$, and the equations reduce to $x = y$. Thus we already conclude that the 1-form $x - y$ vanishes precisely at the fixed points of the involution σ. The equation $xy = zw$ now gives $x^2 = zw$, and the equation

$$x^3 + y^3 = k(z^3 + w^3)$$

becomes

$$2x^3 = k(z^3 + w^3).$$

Multiplying both sides by z^3, we obtain

$$\begin{aligned} 2x^3z^3 =& k(z^6 + w^3z^3) \\ =& k(z^6 + x^6). \end{aligned}$$

In other words, the ratio z/x satisfies a polynomial of degree 6, with no repeated roots. The key point is that since $k \neq \pm 1$, the quadratic above in z^3/x^3 is not a perfect square. There are six distinct values for z/x and, given x and z, the equations $y = x$ and $w = xy/z$ determine y and w. There are six fixed points to the involution σ.

Therefore the map $X \longrightarrow X/\sigma$ is ramified at six points. The Riemann–Hurwitz formula relates the genus of X, denoted $g(X)$, with the genus of X/σ, denoted $g(X/\sigma)$. The general formula says

$$2g(X) - 2 = n[2g(X/\sigma) - 2] + r,$$

where n is the degree of the cover and r is the number of ramification points. In our case, $n = 2$. And we have just computed that $r = 6$. We also know that $g(X) = 4$, and the formula allows us to compute $g(X/\sigma)$; we have

$$6 = 2[2g(X/\sigma) - 2] + 6,$$

giving $g(X/\sigma) = 1$.

Thus X/σ is an elliptic curve, and the map $X \longrightarrow X/\sigma$ is ramified at six points, and those six points are precisely where the 1-form $x - y$ vanishes. This completes the proof of Lemma 3.1. ∎

Remark 3.2: The involutions $\sigma\tau$ and $\sigma\tau^2$ are conjugate to σ in $S_3 \times S_3$. We immediately conclude that the curves $X/\sigma\tau$ and $X/\sigma\tau^2$ are also elliptic, and that the maps $X \longrightarrow X/\sigma\tau$ and $X \longrightarrow X/\sigma\tau^2$ are each ramified at six points. The automorphisms $\overline{\sigma}$, $\overline{\sigma\tau}$ and $\overline{\sigma\tau}^2$ are not conjugate in $S_3 \times S_3$ to the above, but are still very similar. The right way to say it, is that the equations for the curve X can be written

$$\begin{aligned} xy =& zw, \\ k^{-1}(x^3+y^3) =& z^3+w^3, \end{aligned}$$

and that the maps σ and τ for the curve X become the maps $\overline{\sigma}$ and $\overline{\tau}$ for the curve with k replaced by k^{-1}. Hence, applying Lemma 3.1 to the curve with k replaced by k^{-1}, we can conclude that the curves $X/\overline{\sigma}$, $X/\overline{\sigma\tau}$ and $X/\overline{\sigma\tau}^2$ are all elliptic, and the covers $X \longrightarrow X/\overline{\sigma}$, $X \longrightarrow X/\overline{\sigma\tau}$ and $X \longrightarrow X/\overline{\sigma\tau}^2$ each have six ramification points.

LEMMA 3.3: *Since X/σ is an elliptic curve, it has, up to scalar multiples, only one non-zero holomorphic 1-form. Call this 1-form θ. The differential of the map $\phi\colon X \longrightarrow X/\sigma$ takes θ to the 1-form $x-y$ [again, up to scalar multiples].*

Proof: The non-zero holomorphic 1-form θ on X/σ is nowhere vanishing. Its pullback to X is a 1-form, which vanishes precisely at the six points of ramification of the map $\phi\colon X \longrightarrow X/\sigma$. These six ramification points are the fixed points of the involution σ. Now $x-y$ is a 1-form on X (since the embedding in $\mathbb{P}^3$ is canonical), and it vanishes exactly at the fixed points of σ; see Lemma 3.1. The 1-forms $\phi^*\theta$ and $x-y$ have the same sets of zeros. Therefore, up to a scalar multiple, they must agree. ∎

Remark 3.4: There are similar statements, which we can now deduce for $\sigma\tau$, $\sigma\tau^2$, $\overline{\sigma}$, $\overline{\sigma\tau}$ and $\overline{\sigma\tau}^2$. Up to scalar multiples, we have

3.4.1. The holomorphic 1-form on $X/\sigma\tau$ pulls back to $\zeta^2x-\zeta y$.
3.4.2. The holomorphic 1-form on $X/\sigma\tau^2$ pulls back to $\zeta x-\zeta^2 y$.
3.4.3. The holomorphic 1-form on $X/\overline{\sigma}$ pulls back to $w-z$.
3.4.4. The holomorphic 1-form on $X/\overline{\sigma\tau}$ pulls back to $\zeta^2 w-\zeta z$.
3.4.5. The holomorphic 1-form on $X/\overline{\sigma\tau}^2$ pulls back to $\zeta w-\zeta^2 z$.

THEOREM 3.5: *The action of the group $S_3 \times S_3$, on the space of holomorphic 1-forms on X, is given by the formulas*

$$\begin{aligned} \sigma^*(x,y,z,w) =& (-y,-x,-z,-w), \\ \tau^*(x,y,z,w) =& (\zeta x,\zeta^2 y,z,w), \\ \overline{\sigma}^*(x,y,z,w) =& (-x,-y,-w,-z), \\ \overline{\tau}^*(x,y,z,w) =& (x,y,\zeta z,\zeta^2 w). \end{aligned}$$

Proof: The action of $S_3 \times S_3$ on X was defined by the formulas

$$\begin{aligned}\sigma(x,y,z,w) =&(y,x,z,w),\\ \tau(x,y,z,w) =&(\zeta x,\zeta^2 y,z,w),\\ \overline{\sigma}(x,y,z,w) =&(x,y,w,z),\\ \overline{\tau}(x,y,z,w) =&(x,y,\zeta z,\zeta^2 w).\end{aligned}$$

Of course, since the action is on a projective variety, multiplying by a scalar λ has no effect. The points (x,y,z,w) and $\lambda(x,y,z,w)$ agree. So we could, for any $\lambda \in \mathbb{C}^*$, put

$$\sigma(x,y,z,w) = \lambda(y,x,z,w).$$

But in fact, $\{x,y,z,w\}$ are 1-forms on X, and hence so are the pullbacks by σ, denoted $\{\sigma^*x,\sigma^*y,\sigma^*z,\sigma^*w\}$. There is some choice of λ, for which the formula reflects an identity of 1-forms.

To compute the right choice of λ, it is helpful to note that under the map $X \longrightarrow X/\sigma$, the 1-form θ on X/σ pulls back to $x-y$ on X. It follows that the 1-form $x-y$ is invariant under the pullback by σ. But we have

$$\sigma^*(x-y) = \lambda(y-x).$$

This forces $\lambda = -1$, and we deduce that, on the level of 1-forms,

$$\sigma^*(x,y,z,w) = (-y,-x,-z,-w).$$

Similarly, we have that

$$[\sigma\tau]^*(x,y,z,w) = \lambda(\zeta^2 y,\zeta x,z,w).$$

By 3.4.1, the pullback of the holomorphic 1-form on $X/\sigma\tau$ is [up to scalar multiples] $\zeta^2 x - \zeta y$. Whatever the scalar, this means that the 1-form $\zeta^2 x - \zeta y$ is invariant under the automorphism $\sigma\tau$. That is,

$$\begin{aligned}\zeta^2 x - \zeta y =&[\sigma\tau]^*(\zeta^2 x - \zeta y)\\ =&\zeta^2(\lambda\zeta^2 y) - \zeta(\lambda\zeta x)\\ =&\lambda(\zeta y - \zeta^2 x).\end{aligned}$$

This gives $\lambda = -1$, in other words

$$[\sigma\tau]^*(x,y,z,w) = (-\zeta^2 y,-\zeta x,-z,-w).$$

Now we can compute τ; we have

$$\begin{aligned}\tau^*(x,y,z,w) =& [\sigma\tau]^*\sigma^*(x,y,z,w)] \\ =& [\sigma\tau]^*(-y,-x,-z,-w) \\ =& (\zeta x, \zeta^2 y, z, w).\end{aligned}$$

The identities for $\overline{\sigma}$ and $\overline{\tau}$ are obtained by replacing k by k^{-1}; we deduce that the action on the holomorphic differentials is given by the formulas

$$\begin{aligned}\sigma^*(x,y,z,w) =& (-y,-x,-z,-w), \\ \tau^*(x,y,z,w) =& (\zeta x, \zeta^2 y, z, w), \\ \overline{\sigma}^*(x,y,z,w) =& (-x,-y,-w,-z), \\ \overline{\tau}^*(x,y,z,w) =& (x, y, \zeta z, \zeta^2 w). \qquad \blacksquare\end{aligned}$$

Remark 3.6: Note that the action of $S_3 \times S_3$ on the space of $H^0(X, K_X)$ of holomorphic 1-forms is faithful; no element of $S_3 \times S_3$ acts trivially. It follows that each element $g \in S_3 \times S_3$ is a non-trivial automorphism of X; in particular, g has finitely many fixed points. One usually denotes the fixed points of g by X^g. Let $F \subset X$ be the union of all the $\{S_3 \times S_3\}$-orbits of fixed points: in symbols, this says

$$F = \bigcup_{g \in S_3 \times S_3} \{S_3 \times S_3\} X^g.$$

Then F is a finite set and, outside F, the group $S_3 \times S_3$ acts freely.

Let H be a subgroup of $S_3 \times S_3$ and let $N(H)$ be the normalizer of H. The orbit space X/H is a smooth algebraic curve and $N(H)/H$ acts on X/H. By the above, away from a finite set of points in X/H, the action of $N(H)/H$ is free.

Remark 3.7: This concludes our section on the action of $S_3 \times S_3$. We have not only shown there is an action; we have computed what it is, on the vector space $H^0(X, K_X)$ of holomorphic 1-forms.

4. The isogeny

By Lemma 3.1 and Remark 3.2, we know that there are six maps from X to elliptic curves. They are all the projections of X to

$$X/\sigma, \quad X/\sigma\tau, \quad X/\sigma\tau^2, \quad X/\overline{\sigma}, \quad X/\overline{\sigma\tau}, \quad X/\overline{\sigma\tau}^2.$$

In Lemma 3.3 and Remark 3.4 we even computed all the derivatives of these maps. The non-zero holomorphic 1-form on the elliptic curve pulls back, respectively, to

$$x - y, \quad \zeta^2 x - \zeta y, \quad \zeta x - \zeta y^2, \quad z - w, \quad \zeta^2 z - \zeta w, \quad \zeta z - \zeta^2 w.$$

Of course, any map from X to an abelian variety factors through the universal map, to the Jacobian. Let $J = J(X)$ be the Jacobian of X; it maps to the six elliptic curves. And since the map $H^0(J, \Omega^1_J) \longrightarrow H^0(X, K_X)$ is an isomorphism, the above computes the differential of the maps from the Jacobian of X to the six elliptic curves. If we consider the map

$$\Phi\colon J \longrightarrow \{X/\sigma\} \times \{X/\sigma\tau\} \times \{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\},$$

then the differential is clearly an isomorphism. It is enough to check this on global 1-forms, and the obvious basis of global 1-forms on $\{X/\sigma\} \times \{X/\sigma\tau\} \times \{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\}$ pulls back to

$$\{x - y, \zeta^2 x - \zeta y, z - w, \zeta^2 z - \zeta w\},$$

which is a basis for the 1-forms on J. In other words, the map Φ is an isogeny. We next want to study its kernel.

First, let us remind the reader of a useful trick. It is a special case of a much more general fact. But we need only the special case.

LEMMA 4.1: *Let $s\colon Y \longrightarrow Y$ be an involution of a curve Y. Suppose $p \in J(Y)$ is a point on the Jacobian of Y, which lies in the kernel of the natural map*

$$J(Y) \xrightarrow{\pi} J(Y/s).$$

Then $\{1 + s\}(p) = 0$.

Proof: Let $i\colon Y \longrightarrow J(Y)$ be the inclusion of the curve Y in its Jacobian. Consider the map $i + is\colon Y \longrightarrow J(Y)$. It is given by

$$P \mapsto i(P) + is(P),$$

for all $P \in Y$. Clearly $\{i + is\}(P) = \{i + is\}(sP)$, and hence the map $i + is$ factors through Y/s. It can be written

$$Y \longrightarrow Y/s \longrightarrow J(Y).$$

But because the Jacobian has the universal property with respect to maps into abelian varieties, this can be extended to a commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & Y/s & & \\ \downarrow & & \downarrow & & \\ J(Y) & \xrightarrow{\pi} & J(Y/s) & \longrightarrow & J(Y). \end{array}$$

In other words, we have factored the map $1 + s : J(Y) \longrightarrow J(Y)$ as a composite

$$J(Y) \xrightarrow{\pi} J(Y/s) \xrightarrow{\phi} J(Y).$$

If $p \in J(Y)$ is a point so that $\pi(p) = 0$, then certainly $\phi\pi(p) = 0$, that is $\{1+s\}(p) = 0$. ∎

PROPOSITION 4.2: *Let X be the curve of Sections 2 and 3. Let σ, τ, $\overline{\sigma}$ and $\overline{\tau}$ be the standard generators of the $S_3 \times S_3$ acting on X. Then the kernel of the isogeny*

$$\Phi \colon J \longrightarrow \{X/\sigma\} \times \{X/\sigma\tau\} \times \{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}$$

is annihilated by 3. For every element $\rho \in \mathrm{Ker}(\Phi)$, we have $3\rho = 0$. Furthermore, for any element $\rho \in \mathrm{Ker}(\Phi)$, the group $S_3 \times S_3$ acts by a character. Explicitly,

$$\begin{aligned}\sigma(\rho) &= -\rho,\\ \tau(\rho) &= \rho,\\ \overline{\sigma}(\rho) &= -\rho,\\ \overline{\tau}(\rho) &= \rho.\end{aligned}$$

Proof: Suppose ρ is an element of $\mathrm{Ker}(\Phi)$. Then ρ is killed by each of the four maps

$$\begin{aligned}J(X) &\longrightarrow J(X/\sigma), & J(X) &\longrightarrow J(X/\sigma\tau),\\ J(X) &\longrightarrow J(X/\overline{\sigma}), & J(X) &\longrightarrow J(X/\overline{\sigma}\overline{\tau}).\end{aligned}$$

By Lemma 4.1, we conclude that $1+\sigma$, $1+\sigma\tau$, $1+\overline{\sigma}$ and $1+\overline{\sigma}\overline{\tau}$ must all annihilate ρ. This gives the identities

$$\begin{aligned}\sigma(\rho) &= -\rho,\\ \sigma\tau(\rho) &= -\rho,\\ \overline{\sigma}(\rho) &= -\rho,\\ \overline{\sigma}\overline{\tau}(\rho) &= -\rho.\end{aligned}$$

We conclude that

$$\tau(\rho) = \sigma[\sigma\tau(\rho)] = -[-\rho] = \rho,$$

and similarly $\overline{\tau}(\rho) = \rho$. We have therefore already shown the identities

$$\begin{aligned}\sigma(\rho) &= -\rho,\\ \tau(\rho) &= \rho,\\ \overline{\sigma}(\rho) &= -\rho,\\ \overline{\tau}(\rho) &= \rho.\end{aligned}$$

To conclude the proof, we will establish that if $\tau(\rho) = \rho$ and $\overline{\tau}(\rho) = \rho$, then $3\rho = 0$.

For this computation, it is convenient to write $J(X)$ as $\mathbb{C}^4/\Lambda$, for some lattice Λ. Of course, the exponential map from the tangent space of X is a homomorphism, and its kernel is a lattice. Thus $J(X)$ can be expressed as its tangent space modulo a lattice. But $J(X)$ is a principally polarized abelian variety, and hence is canonically isomorphic to its dual, with an isomorphism respecting the $S_3 \times S_3$ action of the automorphisms of X. In other words, $J(X)$ may be written as $H^0(X, K_X)/\Lambda$, for some lattice Λ stable under the $S_3 \times S_3$ action.

Suppose now that $a, b, c, d \in \mathbb{C}$ are complex numbers, and $ax + by + cz + dw \in H^0(X, K_X)$ maps to an element $\rho \in J(X)$, with

$$\tau(\rho) = \rho = \overline{\tau}(\rho).$$

Because ρ is fixed by τ, we have

$$\rho = \tau(\rho) = \tau^2(\rho).$$

This gives two identities:

$$\begin{aligned} ax + by + cz + dw &= a\zeta x + b\zeta^2 y + cz + dw \pmod{\Lambda}, \\ ax + by + cz + dw &= a\zeta^2 x + b\zeta y + cz + dw \pmod{\Lambda}. \end{aligned}$$

In other words, we conclude

$$\begin{aligned} a(1-\zeta)x + b(1-\zeta^2)y &\in \Lambda, \\ a(1-\zeta^2)x + b(1-\zeta)y &\in \Lambda. \end{aligned}$$

Adding these two elements of Λ, we have

$$a(2-\zeta-\zeta^2)x + b(2-\zeta-\zeta^2)y \in \Lambda.$$

But $1 + \zeta + \zeta^2 = 0$, hence $2 - \zeta - \zeta^2 = 3$. We deduce

$$3ax + 3by \in \Lambda.$$

Repeating the argument, with $\overline{\tau}$ in place of τ, we compute that

$$3cz + 3dw \in \Lambda.$$

Adding these two elements of Λ, we have

$$3ax + 3by + 3cz + 3dw \in \Lambda;$$

that is, $3\rho = 0$. ∎

COROLLARY 4.3: *The group $S_3 \times S_3$ acts on the product of elliptic curves*

$$\{X/\sigma\} \times \{X/\sigma\tau\} \times \{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\}$$

in such a way that the isogeny

$$\Phi\colon J \longrightarrow \{X/\sigma\} \times \{X/\sigma\tau\} \times \{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\}$$

is compatible with the $S_3 \times S_3$ action.

Proof: It suffices to show that the kernel of Φ is stable under the $S_3 \times S_3$ action on $J(X)$. But the formulas

$$\begin{aligned} \sigma(\rho) &= -\rho, \\ \tau(\rho) &= \rho, \\ \overline{\sigma}(\rho) &= -\rho, \\ \overline{\tau}(\rho) &= \rho \end{aligned}$$

of Proposition 4.2 tell us that for every $g \in S_3 \times S_3$ and every $\rho \in \mathrm{Ker}(\Phi)$,

$$g(\rho) = \pm\rho.$$

That is, $g(\rho)$ is also an element of $\mathrm{Ker}(\Phi)$. ∎

5. The $S_3 \times S_3$ action on the product

To proceed further, it will be helpful to explicitly compute the $S_3 \times S_3$ action on the product of elliptic curves

$$\Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\}\Big],$$

which in Corollary 4.3 we proved had to exist.

First observe that we know the derivative of the map

$$\begin{array}{ccc} J(X) \xrightarrow{\ \Phi\ } & \Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma\tau}\}\Big] \\ & \Big\downarrow{\scriptstyle \pi_1} \\ & \{X/\sigma\} \times \{X/\sigma\tau\}. \end{array}$$

The codifferential $\Phi^*\pi_1^*$ takes 1-forms on the product of two elliptic curves to 1-forms on J, and the image of the dual Lie algebra of $\{X/\sigma\} \times \{X/\sigma\tau\}$ is spanned by the 1-forms $x - y$ and $\zeta^2 x - \zeta y$. That is, the image is the 2-dimensional vector subspace spanned by $\{x, y\}$.

In Theorem 3.5 we computed, very explicitly, the formulas for the action of $S_3 \times S_3$ on the cotangent space of $J(X)$, which is also the cotangent space of

$$\Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}\Big].$$

It is easily seen that the subspace spanned by the 1-forms $\{x, y\}$ is $S_3 \times S_3$ invariant. It follows that $S_3 \times S_3$ acts on the connected component of the kernels of the maps π_1 and $\pi_1\Phi$. In the case of π_1, the kernel is connected. It is $\{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}$. In other words, we learn that the $S_3 \times S_3$ action on

$$\Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}\Big]$$

carries the kernel of the map

$$\Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}\Big] \xrightarrow{\pi_1} \{X/\sigma\} \times \{X/\sigma\tau\}$$

to itself; there is a unique action of $S_3 \times S_3$ on $\{X/\sigma\} \times \{X/\sigma\tau\}$, which makes π_1 a map of $\{S_3 \times S_3\}$-spaces. Similarly, there is a unique action of $S_3 \times S_3$ on $\{X/\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}$, which makes the map

$$\Big[\{X/\sigma\} \times \{X/\sigma\tau\}\Big] \times \Big[\{X\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}\Big] \xrightarrow{\pi_2} \{X/\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}$$

a morphism of $\{S_3 \times S_3\}$-spaces. We will actually compute the action on $\{X/\sigma\} \times \{X/\sigma\tau\}$. The action on $\{X/\overline{\sigma}\} \times \{X/\overline{\sigma}\overline{\tau}\}$ is given by switching the roles of the two S_3's, and the action on the product of four elliptic curves is the product of these actions. To recapitulate: there is an $S_3 \times S_3$ action on $\{X/\sigma\} \times \{X/\sigma\tau\}$ which makes

$$J(X) \longrightarrow \{X/\sigma\} \times \{X/\sigma\tau\}$$

a morphism of $\{S_3 \times S_3\}$-spaces. We want to compute the action.

Of course, the action on a Lie group is completely determined by the action on its Lie algebra, and we know what $S_3 \times S_3$ does to the Lie algebra of $\{X/\sigma\} \times \{X/\sigma\tau\}$. The formulas are

$$\begin{aligned}
\sigma(x, y) &= (-y, -x), \\
\tau(x, y) &= (\zeta x, \zeta^2 y), \\
\overline{\sigma}(x, y) &= (-x, -y), \\
\overline{\tau}(x, y) &= (x, y).
\end{aligned}$$

From these formulas it is immediately clear that $\overline{\sigma} = -1$ and $\overline{\tau} = 1$. The more subtle problem is to compute σ and τ.

First we want to slightly rewrite the map $X \longrightarrow X/\sigma\tau$. Let β: $X \longrightarrow X/\sigma$ be the natural projection. Consider the composite

$$X \xrightarrow{\tau^2} X \xrightarrow{\beta} X/\sigma.$$

The map is clearly 2–1, since τ is an automorphism and β is a double cover. We assert that $\beta\tau^2$ is nothing other than the canonical projection

$$X \longrightarrow X/\sigma\tau.$$

The point is very simple. We have

$$\begin{aligned} \beta\tau^2[\sigma\tau(P)] &= \beta\sigma\tau^2(P) && \text{since } \tau^2\sigma = \sigma\tau, \\ &= \beta\tau^2(P) && \text{since } \beta\sigma = \beta. \end{aligned}$$

That is, $\beta\tau^2$ is a double cover, which identifies P with $\sigma\tau(P)$. It is just the map $X \longrightarrow X/\sigma\tau$.

We have the map

$$X \xrightarrow{\begin{pmatrix} \beta \\ \beta\tau^2 \end{pmatrix}} \{X/\sigma\} \times \{X/\sigma\}$$

which we identify with the natural map

$$X \longrightarrow \{X/\sigma\} \times \{X/\sigma\tau\}.$$

One can factorize it as

$$X \longrightarrow J(X) \xrightarrow{\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}} \{X/\sigma\} \times \{X/\sigma\}$$

and we know that the map $\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}$ is a morphism of $S_3 \times S_3$-spaces. We have to compute the $S_3 \times S_3$ action on $\{X/\sigma\} \times \{X/\sigma\}$. Let us begin with a useful lemma.

LEMMA 5.1: *The map $\beta + \beta\tau + \beta\tau^2 : X \longrightarrow X/\sigma$ is constant. Put differently, for all $P \in X$, $\beta(P) + \beta\tau(P) + \beta\tau^2(P)$ is independent of P. Rephrasing yet again,*

$$\beta(P - P_0) + \beta\tau(P - P_0) + \beta\tau^2(P - P_0) = 0.$$

Proof: We prove this by computing differentials. The holomorphic 1-form θ on X/σ pulls back, via the map β, to

$$\beta^*\theta = x - y.$$

This is true up to scalars; multiplying θ by a suitable scalar, we can make it exactly true.

But then the pullback of θ under the map $\beta + \beta\tau + \beta\tau^2$ is computed by

$$\begin{aligned}\{\beta + \beta\tau + \beta\tau^2\}^*(\theta) =& \beta^*(\theta) + \tau^*\beta^*(\theta) + \{\tau^*\}^2\beta^*(\theta)\\ =&(x-y) + (\zeta x - \zeta^2 y) + (\zeta^2 x - \zeta y)\\ =&0.\end{aligned}$$

It follows that the map $\beta + \beta\tau + \beta\tau^2$ collapses X to a point in X/σ. ∎

COROLLARY 5.2: *Let B: $J(X) \longrightarrow J(X/\sigma) = X/\sigma$ be the map induced by β, on the Jacobians. Then*

$$B + B\tau + B\tau^2 = 0.$$

Proof: This is just by the functoriality of the Jacobian; another way to say this, is that it is a consequence of the universal property the Jacobian has, for maps into abelian varieties. ∎

Now we are ready to compute the action of $S_3 \times S_3$. We will prove:

THEOREM 5.3: *The action of σ and τ on*

$$\{X/\sigma\} \times \{X/\sigma\tau\} = \{X/\sigma\} \times \{X/\sigma\}$$

is given, respectively, by the matrices

$$\sigma = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since we already know that $\overline{\sigma}$ acts as -1 and $\overline{\tau}$ acts as 1, this computes the action of $S_3 \times S_3$.

Proof: What we need to check is that the maps σ and τ, defined by the matrices above, commute with the projection from $J(X)$. Let us check this for τ first. We need to establish the commutativity of the diagram

$$\begin{array}{ccc} J(X) & \xrightarrow{\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}} & \{X/\sigma\} \times \{X/\sigma\} \\ \tau\downarrow & & \downarrow\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \\ J(X) & \xrightarrow[\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}]{} & \{X/\sigma\} \times \{X/\sigma\} \end{array}$$

But the composite

$$\begin{array}{ccc} J(X) & & \\ \downarrow{\scriptstyle\tau} & & \\ J(X) & \xrightarrow[\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}]{} & \{X/\sigma\} \times \{X/\sigma\} \end{array}$$

is just

$$\begin{pmatrix} B \\ B\tau^2 \end{pmatrix} \tau = \begin{pmatrix} B\tau \\ B\tau^3 \end{pmatrix} = \begin{pmatrix} B\tau \\ B \end{pmatrix},$$

while the composite

$$\begin{array}{ccc} J(X) & \xrightarrow{\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}} & \{X/\sigma\} \times \{X/\sigma\} \\ & & \downarrow{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}} \\ & & \{X/\sigma\} \times \{X/\sigma\} \end{array}$$

is

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B \\ B\tau^2 \end{pmatrix} = \begin{pmatrix} -B - B\tau^2 \\ B \end{pmatrix} = \begin{pmatrix} B\tau \\ B \end{pmatrix},$$

where for the last identity we used $B + B\tau + B\tau^2 = 0$. This establishes that the formula for τ works.

Next we need to check the formula for σ. That amounts to the commutativity of the square

$$\begin{array}{ccc} J(X) & \xrightarrow{\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}} & \{X/\sigma\} \times \{X/\sigma\} \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}} \\ J(X) & \xrightarrow[\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}]{} & \{X/\sigma\} \times \{X/\sigma\} \end{array}$$

But the composite

$$\begin{array}{ccc} J(X) & & \\ \downarrow{\scriptstyle\sigma} & & \\ J(X) & \xrightarrow[\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}]{} & \{X/\sigma\} \times \{X/\sigma\} \end{array}$$

is just

$$\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}\sigma = \begin{pmatrix} B\sigma \\ B\tau^2\sigma \end{pmatrix} = \begin{pmatrix} B\sigma \\ B\sigma\tau \end{pmatrix} = \begin{pmatrix} B \\ B\tau \end{pmatrix}.$$

The last identity is true because $B\sigma = B$, being the canonical map into $J(X/\sigma)$. The composite in the other order, that is

$$\begin{array}{ccc} J(X) & \xrightarrow{\begin{pmatrix} B \\ B\tau^2 \end{pmatrix}} & \{X/\sigma\}\times\{X/\sigma\} \\ & & \Big\downarrow \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ & & \{X/\sigma\}\times\{X/\sigma\} \end{array}$$

is given by the matrix product

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} B \\ B\tau^2 \end{pmatrix} = \begin{pmatrix} B \\ -B - B\tau^2 \end{pmatrix} = \begin{pmatrix} B \\ B\tau \end{pmatrix}.$$

Once again, the last identity relies on $B + B\tau + B\tau^2 = 0$. Anyway, the two composites are equal, so the formula for σ also checks. ∎

6. The maps on the points of order 3

We have a map of $\{S_3 \times S_3\}$-spaces

$$\begin{pmatrix} B \\ B\tau^2 \\ \overline{B} \\ \overline{B}\overline{\tau}^2 \end{pmatrix} : J(X) \longrightarrow \{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2.$$

Let us put

$$\pi = \begin{pmatrix} B \\ B\tau^2 \\ \overline{B} \\ \overline{B}\overline{\tau}^2 \end{pmatrix}.$$

We know that the map π is an isogeny. More precisely, in Proposition 4.2 we computed that its kernel is contained in the points of order 3 on $J(X)$. This means that multiplication by 3, as a map $J(X) \longrightarrow J(X)$, must factor as

$$J(X)\xrightarrow{\pi}\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2 \xrightarrow{\psi} J(X).$$

The map π respects the $S_3 \times S_3$-action, and so does multiplication by 3: $J(X) \longrightarrow J(X)$. It follows that the map ψ also commutes with the action of $S_3 \times S_3$.

It is classical that the composite

$$\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2 \xrightarrow{\psi} J(X) \xrightarrow{\pi} \{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2$$

is also multiplication by 3. Let us adopt the notation that, for any abelian variety A, A_3 stands for the torsion points of order 3. That is,

$$A_3 = \{\rho \in A \mid 3\rho = 0\}.$$

We have a sequence of maps

$$J(X)_3 \xrightarrow{\pi_3} \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2 \xrightarrow{\psi_3} J(X)_3 \xrightarrow{\pi_3} \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2$$

and this sequence is an exact sequence of abelian groups, with an action of $S_3 \times S_3$.

LEMMA 6.1: *The kernel of π_3 is equal to the kernel of π, and the kernel of ψ_3 is equal to the kernel of ψ.*

Proof: Since $\psi\pi = 3$ and $\pi\psi = 3$, both the kernel of π and of ψ are contained in the kernels of multiplication by 3. In symbols:

$$\mathrm{Ker}(\pi) \subset J(X)_3, \quad \mathrm{Ker}(\psi) \subset \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2.$$

An element of order 3 gets killed by the map to a Jacobian, precisely when it is annihilated by the map to the subgroup of elements of order 3. ∎

PROPOSITION 6.2: *In the exact sequence*

$$\{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2 \xrightarrow{\psi_3} J(X)_3 \xrightarrow{\pi_3} \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2$$

the kernel of ψ_3 contains all elements of $\{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2$ of the form

$$\begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}, \quad a \in \{X/\sigma\}_3,\ b \in \{X/\overline{\sigma}\}_3.$$

Proof: By the exactness, one has

$$\mathrm{Im}(\psi_3) = \mathrm{Ker}(\pi_3) = \mathrm{Ker}(\pi).$$

In Proposition 4.2, we saw that for any $\rho \in \mathrm{Ker}(\pi)$, the following identities hold:

$$\begin{aligned} \sigma(\rho) &= -\rho, \\ \tau(\rho) &= \rho, \\ \overline{\sigma}(\rho) &= -\rho, \\ \overline{\tau}(\rho) &= \rho. \end{aligned}$$

These are identities in

$$\mathrm{Im}(\psi_3) = \frac{\{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2}{\mathrm{Ker}(\psi_3)}.$$

These identities mean that, for any $x \in \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2$, the elements

$$\{1+\sigma\}(x),\ \{1-\tau\}(x),\ \{1+\overline{\sigma}\}(x),\ \{1-\overline{\tau}\}(x)$$

all must lie in $\mathrm{Ker}(\psi_3)$. But we have computed all the matrices involved. Our computations of Theorem 5.3 tell us that if x is the vector

$$x = \begin{pmatrix} -a \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2,$$

then $\{1+\sigma\}(x)$ is given by

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right] \begin{pmatrix} -a \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which comes to

$$\begin{pmatrix} -2a \\ a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix},$$

where the identity $-2a = a$ is because $3a = 0$. Similarly, one easily computes that $\{1+\overline{\sigma}\}(y)$, with y being the vector

$$y = \begin{pmatrix} 0 \\ 0 \\ -b \\ 0 \end{pmatrix} \in \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2,$$

comes to

$$\{1+\overline{\sigma}\}(y) \quad = \quad \begin{pmatrix} 0 \\ 0 \\ b \\ b \end{pmatrix}.$$

Therefore, for any $a \in \{X/\sigma\}_3$ and $b \in \{X/\overline{\sigma}\}_3$, the vector

$$\{1+\sigma\}(x) + \{1+\overline{\sigma}\}(y) = \begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}$$

must lie in $\mathrm{Ker}(\psi_3)$. ∎

COROLLARY 6.3: *As above, let* π *be the map*

$$J(X) \xrightarrow{\pi} \{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2.$$

The order of $\mathrm{Ker}(\pi)$ *divides* 3^4.

Proof: By Lemma 6.1, $\mathrm{Ker}(\pi) = \mathrm{Ker}(\pi_3)$. But for π_3 we have the exact sequence

$$J(X)_3 \xrightarrow{\pi_3} \{X/\sigma\}_3^2 \times \{X/\overline{\sigma}\}_3^2 \xrightarrow{\psi_3} J(X)_3.$$

The exact sequence tells us that $\mathrm{Im}(\pi_3) = \mathrm{Ker}(\psi_3)$, and in Proposition 6.2 we produced a subgroup of $\mathrm{Ker}(\psi_3)$ of order 3^4. Thus 3^4 divides

$$|\,\mathrm{Ker}(\psi_3)| = |\,\mathrm{Im}(\pi_3)|,$$

and since

$$|\,\mathrm{Ker}(\pi_3)| \cdot |\,\mathrm{Im}(\pi_3)| = |J(X)_3| = 3^8,$$

we deduce that $|\,\mathrm{Ker}(\pi_3)| = |\,\mathrm{Ker}(\pi)|$ is a divisor of 3^4. ∎

7. The polarization

So far, we have made very little use of the polarization on $J(X)$. Now we are about to change this. The abelian variety $J(X)$ comes with a principal polarization. There is on it an ample divisor Θ, with

$$\deg(\Theta) = \frac{1}{4!}\Theta^4 = 1.$$

The Theta divisor can be thought of as the image of the map

$$X^3 \longrightarrow J(X),$$

given by $(P, Q, R) \mapsto P + Q + R$. Viewed this way, Θ is naturally a divisor in the Jacobian of line bundles of degree 3 on X. Up to non-canonical translation, this can be identified with the Jacobian of degree zero line bundles. For us, the Theta divisor $\Theta \subset J(X)$ is defined only up to non-canonical translation. Its class in the Néron–Severi group $\mathrm{NS}(X)$ is well-defined. Recall that

$$\mathrm{NS}(X) \quad = \quad \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)}$$

is the quotient of the Picard group $\mathrm{Pic}(X)$ by its connected component $\mathrm{Pic}^0(X)$.

This makes it clear that the Theta divisor is invariant under the automorphisms of X [as an element of $\mathrm{NS}(X)$]. Its first Chern class $c_1(\Theta) \in H^2(J(X))$ is invariant under the automorphisms of X.

Our next aim is to identify $c_1(\Theta)$. It lies in

$$H^2(J(X), \mathbb{Q}) \cap H^{1,1}(J(X))$$

and is invariant under the action of $S_3 \times S_3$. It turns out that this almost completely specifies $c_1(\Theta)$. We will prove

LEMMA 7.1: *The $\{S_3 \times S_3\}$-invariant subspace of*

$$H^{1,1}(J(X)) = H^{1,1}(\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2)$$

is a 2-dimensional vector space.

Proof: For the abelian variety $J(X)$, we have

$$H^{1,1}(J(X)) = H^0(X, K_X) \otimes \overline{H^0(X, K_X)}.$$

That is, $H^{1,1}$ is the tensor product $H^{0,1} \otimes \overline{H^{0,1}}$. But, as $S_3 \times S_3$ modules, $\overline{H^0(X, K_X)}$ is the dual of $H^0(X, K_X)$. Thus

$$\begin{aligned} H^{1,1}(J(X)) &= H^0(X, K_X) \otimes \overline{H^0(X, K_X)} \\ &= H^0(X, K_X) \otimes H^0(X, K_X)^* \\ &= \mathrm{End}\left\{H^0(X, K_X)\right\}. \end{aligned}$$

Now the subspace of $\{S_3 \times S_3\}$-invariant elements in $\mathrm{End}\left\{H^0(X, K_X)\right\}$ is the space of maps

$$f\colon H^0(X, K_X) \longrightarrow H^0(X, K_X)$$

commuting with the action of $S_3 \times S_3$. Now we recall that $H^0(X, K_X) = V \oplus W$, with V the vector space spanned by x and y, and W the vector space spanned by w and z. This decomposition respects the action of $S_3 \times S_3$, and V and W are non-isomorphic, irreducible representations of $S_3 \times S_3$. All the statements made above are immediate consequences of the explicit formulas for the action of $S_3 \times S_3$ on $H^0(X, K_X)$.

If follows that any map

$$f\colon V \oplus W \longrightarrow V \oplus W$$

respecting the action of $S_3 \times S_3$ must be of the form $\alpha 1_V + \beta 1_W$. The vector space of all such maps is 2-dimensional. ∎

COROLLARY 7.2: *The rank of the $\{S_3 \times S_3\}$-fixed part of the (rational) Néron–Severi group*

$$\mathrm{NS}(J(X)) = \mathrm{NS}\left(\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2\right)$$

is at most 2.

Proof: The Néron–Severi group is the intersection

$$H^2\big(J(X), \mathbb{Q}\big) \cap H^{1,1}\big(J(X)\big),$$

and since the dimension of the part of $H^{1,1}$ fixed by $S_3 \times S_3$ is only 2, the rank of the fixed part of the Néron–Severi group is bounded above by 2. ∎

Next, we make a useful definition.

Definition 7.3: Let E be an elliptic curve. On $E \times E$, we wish to consider the divisor $D(E) \subset E^2$, given by the sum of three linear subvarieties

$$D(E) = \Big[(x = 0) + (x + y = 0) + (y = 0)\Big].$$

We will prove

LEMMA 7.4: *The rank of the $\{S_3 \times S_3\}$-fixed part of the Néron–Severi group*

$$\mathrm{NS}\left(\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2\right)$$

is exactly 2. *More precisely, this* 2*-dimensional $\mathbb{Q}$-vector space is spanned by the two divisors*

$$\big[D(X/\sigma)\big] \times \{X/\overline{\sigma}\}^2, \quad \{X/\sigma\}^2 \times \big[D(X/\overline{\sigma})\big].$$

Proof: The two divisors are clearly linearly independent; the only problem is to prove that they belong to the fixed part of the Néron–Severi group. We need to show that

$$\big[D(X/\sigma)\big] \subset \{X/\sigma\}^2, \quad \big[D(X/\overline{\sigma})\big] \subset \{X/\overline{\sigma}\}^2$$

are fixed by $S_3 \times S_3$. The two cases being the same, up to replacing k by k^{-1}, we will prove that

$$\big[D(X/\sigma)\big] \subset \{X/\sigma\}^2$$

is fixed by the group.

It suffices to show that $D(X/\sigma)$ is stabilized by the generators σ, τ, $\overline{\sigma}$ and $\overline{\tau}$ of $S_3 \times S_3$. For $\overline{\sigma}$ and $\overline{\tau}$ this is obvious. The involution $\overline{\sigma}$ acts on E^2 by multiplication by -1, while $\overline{\tau}$ acts as the identity. Both maps fix all abelian subvarieties of E^2. But D is the sum of three abelian subvarieties, each of which is fixed.

It remains to study what happens under the action of σ and τ. But the matrices for σ and τ were computed, quite explicitly, in Theorem 5.3. Note that a typical point on the three divisors making up $D(X/\sigma)$ is given by

$$\begin{array}{lll} \text{a point on} & (x=0) & \text{is of the form } \begin{pmatrix} 0 \\ a \end{pmatrix}, \\ \text{a point on} & (x+y=0) & \text{is of the form } \begin{pmatrix} a \\ -a \end{pmatrix}, \\ \text{a point on} & (y=0) & \text{is of the form } \begin{pmatrix} a \\ 0 \end{pmatrix}. \end{array}$$

The matrices for σ and τ are, respectively,

$$\sigma = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and one easily computes that σ takes

$$\begin{array}{llll} \text{the point} & \begin{pmatrix} 0 \\ a \end{pmatrix} & \text{to the point} & \begin{pmatrix} 0 \\ -a \end{pmatrix}, \\ \text{the point} & \begin{pmatrix} a \\ -a \end{pmatrix} & \text{to the point} & \begin{pmatrix} a \\ 0 \end{pmatrix}, \\ \text{the point} & \begin{pmatrix} a \\ 0 \end{pmatrix} & \text{to the point} & \begin{pmatrix} a \\ -a \end{pmatrix}. \end{array}$$

Similarly, one computes that τ takes

$$\begin{array}{llll} \text{the point} & \begin{pmatrix} 0 \\ a \end{pmatrix} & \text{to the point} & \begin{pmatrix} -a \\ 0 \end{pmatrix}, \\ \text{the point} & \begin{pmatrix} a \\ -a \end{pmatrix} & \text{to the point} & \begin{pmatrix} 0 \\ a \end{pmatrix}, \\ \text{the point} & \begin{pmatrix} a \\ 0 \end{pmatrix} & \text{to the point} & \begin{pmatrix} -a \\ a \end{pmatrix}. \end{array}$$

The upshot of the computation is that $D(X/\sigma)$ is fixed, inside $\{X/\sigma\}^2$, by all of $S_3 \times S_3$. ∎

LEMMA 7.5: *Let*

$$\pi\colon J(X) \longrightarrow \{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2$$

be the projection. Let $\Theta \subset J(X)$ be the Theta-divisor. There exists a rational number r so that, up to numerical equivalence,

$$\pi_*\Theta = r\Big[D(X/\sigma) \times \{X/\overline{\sigma}\}^2 + \{X/\sigma\}^2 \times D(X/\overline{\sigma})\Big].$$

Proof: Because the divisors $D(X/\sigma) \times \{X/\overline{\sigma}\}^2$ and $\{X/\sigma\}^2 \times D(X/\overline{\sigma})$ are a basis for the $\{S_3 \times S_3\}$-fixed part of the Néron–Severi group, and because $\pi_*\Theta$ is certainly fixed by $S_3 \times S_3$, there exist unique rational numbers r and s so that

$$\pi_*\Theta = r\Big[D(X/\sigma) \times \{X/\overline{\sigma}\}^2\Big] + s\Big[\{X/\sigma\}^2 \times D(X/\overline{\sigma})\Big].$$

But now consider what happens when one varies the curve. For each $k \notin \{0, \pm 1\}$, we have a curve

$$\begin{aligned} xy &= zw, \\ x^3 + y^3 &= k(z^3 + w^3). \end{aligned}$$

As we vary k, we get a family of curves, a family of Theta-divisors, and a family of identities

$$\pi_*\Theta = r(k)\Big[D(X/\sigma) \times \{X/\overline{\sigma}\}^2\Big] + s(k)\Big[\{X/\sigma\}^2 \times D(X/\overline{\sigma})\Big].$$

But $r(k)$ and $s(k)$ are rational numbers, which vary continuously in the family. They are constant on connected components of the parametrizing variety. Since $\mathbb{C} - \{0, \pm 1\}$ is connected, it follows that $r(k) = r$ is constant, and $s(k) = s$ is constant.

In particular, $r(k) = r(k^{-1})$ and $s(k) = s(k^{-1})$. But replacing k by k^{-1} switches the roles of σ and $\overline{\sigma}$. That is, $r(k) = s(k^{-1})$. This forces $r = s$, and we deduce

$$\pi_*\Theta = r\Big[D(X/\sigma) \times \{X/\overline{\sigma}\}^2 + \{X/\sigma\}^2 \times D(X/\overline{\sigma})\Big]. \qquad \blacksquare$$

In the next Lemma, we compute degrees.

LEMMA 7.6: *Let E be an elliptic curve. The degree of the divisor $D(E) \subset E^2$ is 3.*

Proof: We have, by the definition of $D(E)$,

$$D(E) = \Big[(x = 0) + (x + y = 0) + (y = 0)\Big].$$

But then we can compute the self-intersection of $D(E)$. Each elliptic curve on the abelian surface E^2 has self-intersection 0, and so the self-intersection of $D(E)$

is given by

$$\begin{aligned}D(E)^2 =&2(x=0)\cdot(x+y=0)+2(x+y=0)\cdot(y=0)+2(y=0)\cdot(x=0)\\=&2+2+2\\=&6.\end{aligned}$$

This makes

$$\deg(D(E)) = \frac{1}{2!}D(E)^2 = 3. \qquad \blacksquare$$

COROLLARY 7.7: *An immediate consequence is that the degree of the divisor*

$$D(X/\sigma)\times\{X/\overline{\sigma}\}^2+\{X/\sigma\}^2\times D(X/\overline{\sigma})$$

is $3\times 3=9$. ∎

THEOREM 7.8: *The degree of the isogeny*

$$\pi\colon J(X)\longrightarrow\{X/\sigma\}^2\times\{X/\overline{\sigma}\}^2$$

is precisely 9.

Proof: Let the degree of π be n. The divisor $\Theta\subset J(X)$ has degree 1, and hence $\pi_*\Theta\subset\{X/\sigma\}^2\times\{X/\overline{\sigma}\}^2$ has degree n. But in Lemma 7.5 we saw that, for some rational number r,

$$\pi_*\Theta = r\Big[D(X/\sigma)\times\{X/\overline{\sigma}\}^2+\{X/\sigma\}^2\times D(X/\overline{\sigma})\Big].$$

This makes

$$\begin{aligned}\deg(\pi_*\Theta) =&r^4\deg\Big[D(X/\sigma)\times\{X/\overline{\sigma}\}^2+\{X/\sigma\}^2\times D(X/\overline{\sigma})\Big]\\=&9r^4.\end{aligned}$$

We conclude that $n=9r^4$. Since n is an integer, so is r.

But in Corollary 6.3, we proved that n divides 3^4. Thus for some integer r, $9r^4$ divides 3^4. This can only happen if $r^4=1$, that is $n=9r^4=9$. Thus, the degree of the map π must be 9. ∎

8. A computation on $E \times E$

Let E be an elliptic curve. Consider on $E \times E$ the three divisors, each of which is the sum of abelian subvarieties

$$\begin{aligned} a =&(x=0)+(y=0), \\ b =&(x=0)-(y=0), \\ c =&(x+y=0)-(x=0)-(y=0). \end{aligned}$$

It is very easy to compute the intersections of these divisors. One shows

$$\begin{array}{rcl} a^2 &=& 2 \\ b^2 &=& -2 = c^2 \\ a\cdot b = b\cdot c &=& c\cdot a = 0. \end{array}$$

For any pair of integers $(m,n) \in \mathbb{Z}^2$, let $Z_{m,n}$ be the abelian subvariety $(mx + ny = 0)$. One easily computes the intersection pairings.

LEMMA 8.1:

$$\begin{aligned} (x=0)\cdot Z_{m,n} =& n^2, \\ (y=0)\cdot Z_{m,n} =& m^2, \\ (x+y=0)\cdot Z_{m,n} =& (m-n)^2. \end{aligned}$$

Proof: The divisor $(x=0)$ is the graph of the map $i_2\colon E \longrightarrow E^2$, taking $a \in E$ to $(0,a) \in E^2$. The intersection number $(x=0)\cdot Z_{m,n}$ is the degree of the composite

$$E \xrightarrow{i_2} E^2 \xrightarrow{(m,n)} E.$$

But the composite is multiplication by n, and its degree is n^2.

Similarly, the intersection number $(y=0)\cdot Z_{m,n}$ is the degree of the composite

$$E \xrightarrow{i_1} E^2 \xrightarrow{(m,n)} E.$$

The composite is multiplication by m, and its degree is m^2.

Finally, the intersection number $(x+y=0)\cdot Z_{m,n}$ is the degree of the composite

$$E \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} E^2 \xrightarrow{(m,n)} E.$$

The composite is multiplication by $m-n$, and its degree is $(m-n)^2$. ∎

LEMMA 8.2: *The following identities hold for the intersection numbers:*

$$\begin{aligned} a \cdot Z_{m,n} =& m^2 + n^2, \\ b \cdot Z_{m,n} =& n^2 - m^2, \\ c \cdot Z_{m,n} =& -2mn. \end{aligned}$$

Proof: This is an immediate computation, starting with the identities of Lemma 8.1 and the definitions of a, b and c. Recall

$$\begin{aligned} a =& (x = 0) + (y = 0), \\ b =& (x = 0) - (y = 0), \\ c =& (x + y = 0) - (x = 0) - (y = 0). \end{aligned} \qquad \blacksquare$$

PROPOSITION 8.3: *Up to numerical equivalence, there is an identity*

$$2Z_{m,n} = (m^2 + n^2)a + (m^2 - n^2)b + 2mnc.$$

Proof: Put $W = (m^2 + n^2)a + (m^2 - n^2)b + 2mnc$. One easily computes

$$\begin{aligned} a \cdot W &= 2(m^2 + n^2) &= a \cdot 2Z_{m,n}, \\ b \cdot W &= 2(n^2 - m^2) &= b \cdot 2Z_{m,n}, \\ c \cdot W &= \quad -4mn &= c \cdot 2Z_{m,n}. \end{aligned}$$

Thus

$$(2Z_{m,n} - W) \cdot a = (2Z_{m,n} - W) \cdot b = (2Z_{m,n} - W) \cdot c = 0.$$

Since W is a linear combination of a, b and c, we conclude

$$(2Z_{m,n} - W) \cdot W = 0.$$

That is,

$$\begin{aligned} 2Z_{m,n} \cdot W =& W \cdot W \\ =& \left[(m^2 + n^2)a + (m^2 - n^2)b + 2mnc\right]^2 \\ =& 2(m^2 + n^2)^2 - 2(m^2 - n^2)^2 - 2(2mn)^2 \\ =& 0. \end{aligned}$$

This makes

$$\begin{aligned} (2Z_{m,n} - W)^2 =& 4Z_{m,n}^2 - 4Z_{m,n} \cdot W + W^2 \\ =& 0 + 0 + 0 \\ =& 0. \end{aligned}$$

The identities $2Z_{m,n} \cdot W = W \cdot W = 0$ are the previous computation, and the identity $Z_{m,n}^2 = 0$ is because $Z_{m,n}$ is an abelian subvariety.

Up to now, we have computed intersection numbers, with the upshot that $(2Z_{m,n} - W)^2 = a \cdot (2Z_{m,n} - W) = 0$. But $a^2 = 2 > 0$, and the Hodge index theorem tells us that on the subspace of divisors F satisfying $a \cdot F = 0$, the intersection product is negative definite. The divisor $(2Z_{m,n} - W)$ lies in the subspace, as $a \cdot (2Z_{m,n} - W) = 0$. Since $(2Z_{m,n} - W)^2 = 0$, the Hodge index theorem says $2Z_{m,n} - W$ must vanish, up to numerical equivalence. ∎

PROPOSITION 8.4: *Up to numerical equivalence, the divisor $D(E) \subset E^2$ of Definition 7.3 satisfies the identity*

$$14D(E) = 3Z_{3,1} + Z_{1,5}.$$

Proof: We have

$$\begin{aligned} 2Z_{3,1} =& (3^2 + 1^2)a + (3^2 - 1^2)b + 2 \cdot 3 \cdot 1c \\ =& 10a + 8b + 6c. \end{aligned}$$

We also have

$$\begin{aligned} 2Z_{1,5} =& (1^2 + 5^2)a + (1^2 - 5^2)b + 2 \cdot 1 \cdot 5c \\ =& 26a - 24b + 10c. \end{aligned}$$

This makes

$$\begin{aligned} 6Z_{3,1} + 2Z_{1,5} =& 3(10a + 8b + 6c) + (26a - 24b + 10c) \\ =& 56a + 28c; \end{aligned}$$

in other words

$$3Z_{3,1} + Z_{1,5} = 14(2a + c).$$

But

$$\begin{aligned} 2a + c =& 2\Big[(x = 0) + (y = 0)\Big] + \Big[(x + y = 0) - (x = 0) - (y = 0)\Big] \\ =& \Big[(x + y = 0) + (x = 0) + (y = 0)\Big] \\ =& D(E), \end{aligned}$$

and we conclude

$$3Z_{3,1} + Z_{1,5} = 14D(E). \quad ∎$$

Remark 8.5: There is nothing unique about the above relation. We tried to indicate that there are many relations among the $Z_{m,n}$'s. We will return to study the non-uniqueness in Section 9.

COROLLARY 8.6: *Some suitably large integral multiple of the Theta divisor on $J(X)$ can be written as a positive linear combination of exactly four divisors, each of which is the kernel of some map $J(X) \longrightarrow E$, for some elliptic curve E.*

Proof: Let us do the computation on $\{X/\sigma\}^2 \times \{X/\overline{\sigma}\}^2$, which is isogenous to $J(X)$. The Theta divisor is, up to a scalar,

$$D(X/\sigma) \times \{X/\overline{\sigma}\}^2 + \{X/\sigma\}^2 \times D(X/\overline{\sigma}).$$

But we have just shown that every $D(E)$ is a positive linear combination of two abelian subvarieties. It follows that Θ is a positive linear combination of four. We already remarked that the decomposition is not unique. ∎

Remark 8.7: For each integer $n > 0$, let $n^*\Theta$ denote the pullback of Θ by multiplication by n. The above means that, for some suitably large n, the pullback to $\mathbb{C}^4$ of the divisor $n^*\Theta$ is the vanishing of a function, which can be written as a sum of products of four 1-dimensional Theta functions. In other words, the formula obtained by Matveev and Smirnov in Section 5 of [12] is not surprising. It is one of many, expressing the Theta function of X in a similar-looking form.

9. The non-uniqueness of the identities for divisor classes

In Section 8, we proved some identities of divisor classes on E^2, which permitted us to establish a formula

$$3Z_{3,1} + Z_{1,5} = 14D(E).$$

In Remark 8.5 we noted that this expression is not unique. We wish to study the relations

$$\alpha Z_{m,n} + \beta Z_{x,y} = \gamma D(E),$$

where α, β and γ are rational numbers. In Proposition 8.3, we proved identities

$$\begin{aligned} 2Z_{m,n} =& (m^2 + n^2)a + (m^2 - n^2)b + 2mnc, \\ 2Z_{x,y} =& (x^2 + y^2)a + (x^2 - y^2)b + 2xyc. \end{aligned}$$

We also know that $D(E) = 2a + c$. If $\alpha Z_{m,n} + \beta Z_{x,y} = \gamma D(E)$, then the three vectors $Z_{m,n}$, $Z_{x,y}$ and $D(E)$ must be linearly dependent, in the vector space with basis $\{a, b, c\}$. This leads us to the following Lemma.

LEMMA 9.1: *The three vectors*

$$\begin{aligned} 2Z_{m,n} =&(m^2+n^2)a+(m^2-n^2)b+2mnc, \\ 2Z_{x,y} =&(x^2+y^2)a+(x^2-y^2)b+2xyc, \\ D(E) =&2a+0b+c \end{aligned}$$

are linearly dependent, precisely when the 2×2 *determinant below vanishes; that is, precisely when*

$$\begin{vmatrix} m^2-4mn+n^2 & m^2-n^2 \\ x^2-4xy+y^2 & x^2-y^2 \end{vmatrix} = 0.$$

Proof: The three vectors are linearly dependent, precisely when the 3×3 determinant

$$\begin{vmatrix} m^2+n^2 & m^2-n^2 & 2mn \\ x^2+y^2 & x^2-y^2 & 2xy \\ 2 & 0 & 1 \end{vmatrix}$$

vanishes. But now, expanding along the third row, the determinant becomes

$$\begin{aligned} =&2\begin{vmatrix} m^2-n^2 & 2mn \\ x^2-y^2 & 2xy \end{vmatrix} + \begin{vmatrix} m^2+n^2 & m^2-n^2 \\ x^2+y^2 & x^2-y^2 \end{vmatrix} \\ =&-2\begin{vmatrix} 2mn & m^2-n^2 \\ 2xy & x^2-y^2 \end{vmatrix} + \begin{vmatrix} m^2+n^2 & m^2-n^2 \\ x^2+y^2 & x^2-y^2 \end{vmatrix} \\ =&\begin{vmatrix} m^2-4mn+n^2 & m^2-n^2 \\ x^2-4xy+y^2 & x^2-y^2 \end{vmatrix}. \end{aligned}$$ ∎

LEMMA 9.2: *The two vectors*

$$\begin{aligned} 2Z_{m,n} =&(m^2+n^2)a+(m^2-n^2)b+2mnc, \\ 2Z_{x,y} =&(x^2+y^2)a+(x^2-y^2)b+2xyc \end{aligned}$$

are linearly dependent, if and only if (m,n) *and* (x,y) *are linearly dependent.*

Proof: The "if" part is obvious. If the vectors (m,n) and (x,y) are the same to within scalar multiples, then so are $Z_{m,n}$ and $Z_{x,y}$. We need to prove the converse.

Suppose therefore that

$$\begin{aligned} 2Z_{m,n} =&(m^2+n^2)a+(m^2-n^2)b+2mnc, \\ 2Z_{x,y} =&(x^2+y^2)a+(x^2-y^2)b+2xyc \end{aligned}$$

are linearly dependent. Then one of the vectors is a multiple of the other; interchanging them if necessary, assume $Z_{x,y}=\lambda^2 Z_{m,n}$. From the identities

$$\begin{aligned} x^2+y^2 =&\lambda^2(m^2+n^2), \\ x^2-y^2 =&\lambda^2(m^2-n^2) \end{aligned}$$

we easily deduce that $x^2 = \lambda^2 m^2$ and $y^2 = \lambda^2 n^2$. This gives

$$x = \varepsilon\lambda m, \quad y = \eta\lambda n$$

with $\varepsilon = \pm 1$, $\eta = \pm 1$. But we also have an identity

$$2xy = \lambda^2(2mn).$$

Putting $x = \varepsilon\lambda m$ and $y = \eta\lambda n$, this becomes

$$(\varepsilon\lambda m)(\eta\lambda n) = \lambda^2 mn,$$

which amounts to

$$\lambda^2 mn(\varepsilon\eta - 1) = 0.$$

Therefore either $\lambda = 0$, or $m = 0$, or $n = 0$, or $\varepsilon\eta = 1$. And in every case, the vectors (m, n) and (x, y) are linearly dependent. ∎

LEMMA 9.3: *When the two vectors*

$$\begin{aligned} 2Z_{m,n} =& (m^2 + n^2)a + (m^2 - n^2)b + 2mnc, \\ 2Z_{x,y} =& (x^2 + y^2)a + (x^2 - y^2)b + 2xyc \end{aligned}$$

are linearly dependent, then $D(E)$ does not lie in the linear span of $Z_{m,n}$ and $Z_{x,y}$.

Proof: Since the vectors are linearly dependent, for $D(E)$ to lie in the linear span it would have to be a multiple of one of $Z_{m,n}$ or $Z_{x,y}$. Without loss, assume $D(E) = \alpha Z_{m,n}$. This gives identities

$$\begin{aligned} 2 =& \alpha(m^2 + n^2), \\ 0 =& \alpha(m^2 - n^2), \\ 1 =& \alpha(2mn). \end{aligned}$$

The third equation says $\alpha \neq 0$, and the second says $m^2 = n^2$. The first equation becomes $m^2 = 1/\alpha$, while the third says $m^2 = \pm 1/2\alpha$. These are incompatible, and there is no solution. ∎

Remark 9.4: Lemma 9.3 tells us that when

$$\begin{aligned} 2Z_{m,n} =& (m^2 + n^2)a + (m^2 - n^2)b + 2mnc, \\ 2Z_{x,y} =& (x^2 + y^2)a + (x^2 - y^2)b + 2xyc \end{aligned}$$

are linearly dependent, then $D(E)$ does not lie in the linear span of $Z_{m,n}$ and $Z_{x,y}$. Thus $D(E)$ will lie in the linear span of $Z_{m,n}$ and $Z_{x,y}$ precisely when the vectors

$$D(E), \quad Z_{m,n}, \quad Z_{x,y}$$

are linearly dependent, but the vectors $Z_{m,n}$ and $Z_{x,y}$ are not. By Lemma 9.1, the three vectors

$$\begin{aligned} 2Z_{m,n} &= (m^2+n^2)a + (m^2-n^2)b + 2mnc, \\ 2Z_{x,y} &= (x^2+y^2)a + (x^2-y^2)b + 2xyc, \\ D(E) &= 2a + 0b + c \end{aligned}$$

are linearly dependent precisely when

$$\begin{vmatrix} m^2-4mn+n^2 & m^2-n^2 \\ x^2-4xy+y^2 & x^2-y^2 \end{vmatrix} = 0.$$

By Lemma 9.2, the two vectors

$$\begin{aligned} 2Z_{m,n} &= (m^2+n^2)a + (m^2-n^2)b + 2mnc, \\ 2Z_{x,y} &= (x^2+y^2)a + (x^2-y^2)b + 2xyc \end{aligned}$$

are linearly dependent precisely when (m,n) and (x,y) are linearly dependent. Thus $D(E)$ lies in the linear span of $Z_{m,n}$ and $Z_{x,y}$ precisely when (m,n) and (x,y) are linearly independent, and

$$\begin{vmatrix} m^2-4mn+n^2 & m^2-n^2 \\ x^2-4xy+y^2 & x^2-y^2 \end{vmatrix} = 0.$$

Next we wish to fix a vector $(m,n) \neq (0,0)$ and study what happens for different (x,y). Put $r = m^2-4mn+n^2$ and $s = m^2-n^2$. The vector $D(E)$ will lie in the linear span of $Z_{m,n}$ and $Z_{x,y}$, precisely if (x,y) is not a multiple of (m,n), and

$$\begin{vmatrix} r & s \\ x^2-4xy+y^2 & x^2-y^2 \end{vmatrix} = 0.$$

This equation expands to

$$(r-s)x^2 + 4sxy - (r+s)y^2 = 0.$$

It always has a solution $y/x = n/m$. We want to know if the other (necessarily rational) solution y/x is distinct. In any case, the discriminant is a perfect square. That is, we know

$$[4s]^2 + 4(r-s)(r+s) = 4[3s^2+r^2]$$

to be a perfect square, since the equation has rational solutions. Substituting again $r = m^2 - 4mn + n^2$ and $s = m^2 - n^2$, we compute that

$$\begin{aligned} 3s^3 + r^2 =& 3[m^2 - 4mn + n^2]^2 + [m^2 - n^2]^2 \\ =& 4[m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4] \\ =& 4[m^2 - mn + n^2]^2. \end{aligned}$$

And the interesting observation is that the discriminant is non-zero; the two roots y/x are distinct.

PROPOSITION 9.5: *Suppose we are given some vector $(m, n) \neq (0, 0)$, with $m, n, \in \mathbb{Z}$. There is, up to scalar multiples, exactly one vector $(x, y) \in \mathbb{Z}^2$, so that $D(E)$ lies in the linear span of $Z_{m,n}$ and $Z_{x,y}$. Up to scalars, there is a formula*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Proof: Except for the formula for y/x, everything has already been established. We prove the formula.

By the above, y/x satisfies the quadratic equation

$$[y/x]^2 - \frac{4s}{s+r}[y/x] + \frac{s-r}{s+r} = 0.$$

This equation has two roots, one of which is $y/x = n/m$, and the other is of interest. But we have

$$[y/x] + [n/m] = \frac{4s}{r+s}.$$

From this we compute

$$\begin{aligned} [y/x] =& \frac{4s}{r+s} - [n/m] \\ =& \frac{4[m^2 - n^2]}{2m^2 - 4mn} - [n/m] \\ =& \frac{2m - n}{m - 2n}. \end{aligned}$$

In other words, up to scalar multiples, $x = m - 2n$ and $y = 2m - n$. The fractional linear transformation taking (m, n) to (x, y) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}. \quad \blacksquare$$

In particular, there is a plethora of solutions. There is nothing unique or canonical about the expression, obtained by Matveev and Smirnov's [12], which gave the Theta function as a sum of products of four 1-dimensional Theta functions. For any integers $(m,n) \neq (0,0)$, there is an (x,y) so that $D(E)$ is in the linear span of $Z_{m,n}$ and $Z_{x,y}$.

References

[1] G. E. Andrews, R. J. Baxter and P. J. Forrester, *Eight-vertex SOS model and generalized Rogers–Ramanujan type identities*, Journal of Statistical Physics **35** (1984), 193–266.

[2] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York, 1982.

[3] R. J. Baxter, *Chiral Potts model: Corner transfer matrices and parametrizations*, International Journal of Modern Physics B **7** (1991), 3489–3500.

[4] R. J. Baxter, *Hyperelliptic function parametrization for the chiral Potts model*, Proceedings of the International Congress of Mathematicians, Springer-Verlag, Berlin, 1991, pp. 1305–1317.

[5] R. J. Baxter, *Elliptic parametrization of the three-state chiral Potts models*, in *Integrable Quantum Field Theories*, Plenum Press, New York, 1993, pp. 27–37.

[6] R. J. Baxter, *Functional relations for the order parameters of the chiral Potts model*, Journal of Statistical Physics **91** (1998), 499–524.

[7] R. J. Baxter, *Functional relations for the order parameters of the chiral Potts model: low temperature expansions*, Physica A **260** (1998), 117–130.

[8] R. J. Baxter, *Some hyperelliptic function identities that occur in the chiral Potts model*, Journal of Physics. A. Mathematics and General **31** (1998), 6807–6818.

[9] R. J. Baxter, J. H. H. Perk and H. Au-Yang, *New solutions of the star–triangle relations for the chiral Potts model*, Physics Letters A **128** (1988), 138–142.

[10] B. Davies, *On the Riemann surface of the three-state chiral Potts model*, Statistical Physics on the Eve of the 21st Century, Series of Advances in Statistical Mechanics, Vol. 14, World Scientific, Singapore, 1999, pp. 316–335.

[11] M. Jimbo, T. Miwa and M. Okado, *Solvable lattice models related to the vector representation of classical simple Lie algebras*, Communications in Mathematical Physics **116** (1988), 507–525.

[12] V. B Matveev and A. O. Smirnov, *Some comments on the solvable chiral Potts model*, Letters in Mathematical Physics **19** (1990), 179–185.

[13] L. Onsager, *Crystal statistics I. A two-dimensional model with an order–disorder transition*, Physical Review **65** (1944), 117–149.